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Analytic network process in risk assessment and decision analysis ☆

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ABSTRACT

In risk assessment and decision analysis, the analytical network process (ANP) is widely used to assess the key factors of risks and analyze the impacts and preferences of decision alternatives. There are lots of comparison matrices for a complicated risk assessment problem, but a decision has to be made rapidly in emergency cases. However, in the ANP, the reciprocal pairwise comparison matrices (RPCM) are more complicated and difficult than AHP. The consistency test and the inconsistent elements identification need to be simplified. In this paper, a maximum eigenvalue threshold is proposed as the consistency index for the ANP in risk assessment and decision analysis. The proposed threshold is mathematically equivalent to the consistency ratio (CR). To reduce the times of consistency test, a block diagonal matrix is introduced for the RPCM to conduct consistency tests simultaneously for all comparison matrices. Besides, the inconsistent elements can be identified and adjusted by an induced bias block diagonal comparison matrix. The effectiveness and the simplicity of the proposed maximum eigenvalue threshold consistency test method and the inconsistency identification and adjustment method are shown by two illustrative examples of emergent situations.

1. Introduction

Over the past few decades, risk assessment and decision analysis has been an active research area (for instance: [33,4,13,11,18,24,53,54,55,56,48,20,50,28,29,30,31]). The decision analysts have to make quick and efficient decision for multi-criteria decision making (MCDM) problems such as identifying the key factors of the risk and the potential risk, determining risk level and risk consequences, analyzing the uncertain variables of a decision, considering different preferences, etc. For instance, the emergency managers have to select emergency prevention alternatives, emergency pre-response plans, emergency response alternatives, and emergency recovery alternatives [14].

The AHP (analytical hierarchy process), as a widely used MCDM method, is often implemented in the Benefit–Opportunity–Cost–Risk (BOCR) analysis to improve the effectiveness of risk assessment and decision analysis [52,46,1]. There are several assumptions when the AHP is applied to make decisions, such as, the independence between higher level elements and lower level elements, the independence of the elements within a level, and the hierarchy structure of the decision problem [38,47]. However, in reality, risk assessment and decision analysis problems are often too complicated to be structured hierarchically. In addition, the interactions of decision attributes within the same level and the feedbacks between two different levels are important issues that should be considered during the decision making process. Therefore, the AHP method does not work accurately when solving such decision problems [39].

The analytical network process (ANP), as an extensive and complementary method of the AHP, was introduced and further developed by Saaty [39,40,41,42,43,44,45,46]. The ANP method can be used to make decision problems that cannot be structured hierarchically and does not have the inner-independent and outer-independent assumptions. Since its introduction, the ANP method is applied to diverse areas. For instance, Lee and Kim [21] suggest an improved information system (IS) project selection method using the ANP within a zero–one goal programming model to solve the IS project selection problems. Hafeez et al. [10] provide a structured framework for determining the key capabilities of a firm using the ANP. Karsak et al. [16] employ the ANP to evaluate the interrelationships among customer needs and product technical requirements (PTRs) while determining the importance levels of PTRs in the house of quality (HOQ).

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Table 1
The average random index.

n	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.52	0.89	1.11	1.25	1.35	1.4	1.45	1.49

Niemira and Saaty [27] develop an imbalance-crisis turning point model to forecast the likelihood of a financial crisis based on an ANP framework. Chung et al. [7] employ the ANP to select and evaluate different product mixes in a semiconductor fabricator. Kahraman et al. [17] employ the ANP to obtain the coefficients of the objective function and propose a fuzzy optimization model for Quality function deployment (QFD) planning process using the ANP. Bayazit and Karpak [3] develop an ANP based framework to assess the implementation of total quality management (TQM). Aktar and Ustun [2] suggest an integrated approach of Archimedean Goal Programming (AGP) and Analytic Network Process (ANP) to evaluate the suppliers and determine their periodic shipment allocations given a number of tangible and intangible criteria. Caballero-Luque et al. [5] present a model based on the ANP to help organization managers to verify if their website contents are appropriate for satisfying the goals they have established.

In addition to the above fields, the ANP has also widely been used in risk assessment and decision analysis. For instance, Lu et al. [26] apply the ANP to deal with the degree of risk for the main activities of an urban bridge project. Raisinghani et al. [32] utilize the ANP to provide insight into optimum-seeking decision processes by managers and studies the 'systems with feedback' where the e-commerce strategy may both dominate and be dominated directly and indirectly by the business-level strategy. Dagdeviren et al. [8] employ the ANP to identify the faulty behavior risk (FBR) that is significant in work system safety. Besides, Levy and Taji [22] propose a Group Analytic Network Process (GANP) approach to support hazards planning and emergency management under incomplete information.

In the ANP, similar to the AHP, two issues need to be solved for a reciprocal pairwise comparison matrix (RPCM hereinafter): consistency test and inconsistent element(s) identification and adjustment. However, these issues are more complicated in the ANP than in the AHP since there are more comparison matrices in the ANP. There exist many studies in consistency test [35,34,12,36,25,51,23,6,15,19]. The most well-known consistency index for the pairwise comparison matrices in the AHP/ANP is the consistency ratio by Saaty. That is, $CR = CI/RI < 0.1$, where the consistency index $CI = (\lambda_{\max} - n)/(n-1)$, RI is the average Random Index based on matrix size n, and $\lambda_{\max}$ is the maximum eigenvalue of matrix A [36]. If the CR is less than 0.1, the comparison matrix passes the consistency test. Otherwise, the entry in the corresponding comparison matrix should be revised. For the inconsistent elements identification method, it does not have a commonly accepted method for the AHP/ANP. In the AHP software *Expert Choice*, an error matrix $\varepsilon_{ij} = a_{ij}(w_j/w_i)$ is constructed to identify the most inconsistent element [42].

When the ANP is applied to assess and analyze the factors of the existing risk and potential risks, as well as the impacts of a decision for an emergent event, the consistency of the comparison matrix and the inconsistent elements should be identified and adjusted as soon as possible. The risk assessment and decision analysis of an emergent event is a typical time-critical information service that is highly dependent on time and information. To improve the efficiency of response decision making in risk assessment and decision analysis, in this paper, a maximum eigenvalue threshold index method is proposed as the new consistency index for the ANP. A bias block diagonal matrix consisting of the inconsistent comparison matrices is introduced to rapidly identify and adjust the inconsistent elements in the original inconsistent comparison matrices when the ANP is applied to the risk assessment and decision analysis.

The remaining of this paper is organized as follows. Section 2 briefly reviews the traditional consistency test method and analyzes the issues of consistency test in the ANP. In Section 3, a maximum eigenvalue threshold method is introduced as the consistency index for the ANP. A block diagonal matrix is introduced to test the consistency of all comparison matrices simultaneously. The judgment process of the proposed method is also presented. An induced bias block diagonal matrix for inconsistency identification and adjustment is also proposed. In Section 4, two numeric examples are used to illustrate the proposed consistency index and the inconsistency identification method. Section 5 concludes the paper and discusses future research directions.

2. Existing consistency test method for the ANP

2.1. Existing consistency test method

The risk assessment and decision analysis is often complicated in nature and there are inconsistency issues when different attributes or criteria in the process of risk assessment and decision analysis are compared. 'Inconsistency itself is important because without it, new knowledge that changes preference cannot be admitted' [46, p. 15]. The inconsistencies can be classified into two types, cardinal inconsistency and ordinal inconsistency. For instance, suppose attribute A is 2 times important as attribute B, and attribute B is 3 times important as attribute C; however, attribute A is only 4 times important as attribute C instead of 6 times. Likewise, the values of A is bigger than B, B is bigger than C, and however C is bigger than A, namely, $A > B$, $B > C$, but $C > A$. Both of these issues are called inconsistency [36]. The final risk assessment and decision analysis could be inaccurate if the priority vectors are calculated from the inconsistent comparison matrix. Therefore, the consistency of each comparison matrix has to be tested before the comparison matrices are used to assess risk and analyze a decision. If the consistency test for the comparison matrix is failed, the inconsistent elements in the comparison matrix have to be identified and revised; otherwise, the result of risk assessment and decision analysis is unreliable.

The traditional consistency test method, which is also the most widely used consistency index, is the consistency ratio (CR) [36]:

$$CR = \frac{CI}{RI} < 0.1 \tag{2.1.1}$$

where $CI = (\lambda_{\max} - n)/(n-1)$is the consistency index, RI is the average random index based on Matrix Size shown in Table 1, $\lambda_{\max}$is the maximum eigenvalue of matrix A, and n is the order of matrix A.

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$$W_{ij} \ \textit{Component of Supermatrix}$$

$$W_{ij} = \begin{bmatrix} W_{i1}^{(j_1)} & W_{i1}^{(j_2)} & \cdots & W_{i1}^{(j_{n_j})} \\ W_{i2}^{(j_1)} & W_{i2}^{(j_2)} & \cdots & W_{i2}^{(j_{n_j})} \\ \vdots & \vdots & & \vdots \\ W_{in_i}^{(j_1)} & W_{in_i}^{(j_2)} & \cdots & W_{in_i}^{(j_{n_j})} \end{bmatrix}$$

Fig. 1. The supermatrix of a network.

According to the CR method, the comparison matrix is a consistent comparison matrix only if the value of CR is acceptable (less than 0.1). The CR method includes the following four steps:

Step 1: Calculate the $\lambda_{\max}$ of one comparison matrix.
Step 2: Calculate the value of CI using the formula$CI = (\lambda_{\max} - n)/(n-1)$.
Step 3: Calculate the CR using the formula $CR = CI/RI$ and Table 1.
Step 4: Compare the value of CR with the consistency threshold 0.1 to judge whether the comparison is consistent.

Therefore, there is a major shortcoming when using CR as the consistency index for comparison matrices, as the above steps have to be calculated repeatedly for each comparison matrix to test the consistency.

2.2. *Analysis of the consistency test issues in the ANP*

The CR method is a widely used consistency test method in both AHP and ANP. In the ANP, if a comparison matrix passes the consistency test, the priorities derived from the comparison matrix is added as parts of the columns of the supermatrix of a network [46], which is shown in Fig. 1. Otherwise, this comparison matrix has to be revised by experts. Therefore, the consistency tests will be much more complicated in the ANP than in the AHP since there are more comparison matrices in the ANP, which can be derived from the following supermatrix of a network [46]:

$$W = \begin{array}{cc} & \\ & \\ C_1 & \begin{array}{c} e_{11} \\ e_{12} \\ \vdots \\ e_{1n_1} \end{array} \\ C_2 & \begin{array}{c} e_{21} \\ e_{22} \\ \vdots \\ e_{2n_2} \end{array} \\ & \vdots \\ C_N & \begin{array}{c} e_{N1} \\ e_{N2} \\ \vdots \\ c_{N_iN} \end{array} \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_N \\ e_{11}e_{12}\cdots e_{1n_1} & e_{21}e_{22}\cdots e_{2n_2} & & e_{N1}e_{N2}\cdots e_{Nn_N} \end{array} \\ \begin{bmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ W_{N1} & W_{N2} & \cdots & W_{NN} \end{bmatrix} \end{array}$$

In Fig. 1, each judgment indicates the dominance of an element in the column on the left over an element in the row on the top. Assume $W_{ij} \neq 0$ for all $1 \leq i, j \leq N$, both inner-clusters and outer-clusters have interactions. In the above supermatrix, there are two kinds of comparison matrices in the ANP: the inner-clusters comparison matrices and the outer-clusters comparison matrices. From the C_1 cluster to the C_N cluster, the number of the comparison matrices in the inner-cluster is n_1 with order n_1, n_2 with order n_2,... and n_N with order n_N, respectively. The number of the comparison matrices in the outer-cluster includes: $(N-1)n_1$ with orders $n_2, n_3, \ldots, n_N$; $(N-1)n_2$ with orders $n_1, n_3, \ldots, n_N$; $(N-1)n_3$with orders $n_1, n_2, n_4, \ldots, n_N; \ldots$; $(N-1)n_N$ with orders $n_1, n_2, n_3, \ldots, n_{N-1}$.

Inner−cluster comparison matrices in the ANP

Quantity	**Matrices order**
n_1	$n_1 \times n_1$
n_2	$n_2 \times n_2$
$\vdots$	$\vdots$
n_N	$n_N \times n_N$

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The total number of inner-cluster comparison matrices is $n_1+n_2+\cdots+n_N$

Outer−cluster comparison matrices in the ANP

Quantity	Matrices order
$(N-1)n_1$	$n_2\times n_2, n_3\times n_3, \ldots, n_N\times n_N$
$(N-1)n_2$	$n_1\times n_1, n_3\times n_3, \ldots, n_N\times n_N$
$\vdots$	$\vdots$
$(N-1)n_N$	$n_1\times n_1, n_3\times n_3, \ldots, n_{N-1}\times n_{N-1}$

The total number of outer-cluster comparison matrices is$(N-1)(n_1+n_2+\cdots+n_N)$.

Therefore, the total number of all the comparison matrices in the ANP is$N(n_1+n_2+\cdots+n_N)$. Hence, the CR method has to calculate the consistency ratios $N(n_1+n_2+\cdots+n_N)$ times for all comparison matrices.

To sum up:

(1) The consistency ratio is calculated repeatedly for each comparison matrix in the CR method.
(2) The CRs of the comparison matrices in the ANP need to be calculated $4N(n_1+n_2+\cdots+n_N)$ times since the total number of the comparison matrices is $N(n_1+n_2+\cdots+n_N)$ from C_1 cluster to C_N cluster, which contain n_1 elements to n_N elements, respectively.

3. The new consistency index and inconsistency identification method for the ANP

3.1. The new consistency index for the ANP

In the CR method, the formula $CR=(\lambda_{\max}-n)((n-1)RI)$ has three parameters$\lambda_{\max}$, n, and RI, where $\lambda_{\max}$is the maximum eigenvalue of one comparison matrix, n is the size of the comparison matrix, and RI is the average random index based on matrix size. The CR is only impacted by the $\lambda_{\max}$ for the comparison matrices with the same order. Therefore, the following corollary can be derived:

Corollary. *The inequality CR < 0.1 is mathematically equivalent to the inequality* $\lambda_{\max}<\lambda^n_{\max\ \text{threshold}}$ *or* $\Delta\lambda_{\max}<0$ *, where* $\lambda_{\max}$ *denotes the maximum eigenvalue of the comparison matrix with order n,* $\lambda^n_{\max\ \text{threshold}}$ *represents the corresponding maximum eigenvalue threshold with order n, which is listed in Table 2, and* $\Delta\lambda_{\max}$ *denotes the bias between the maximum eigenvalue and its corresponding threshold.*

Proof. If

$$CR=\frac{CI}{RI}=\frac{\lambda_{\max}-n}{(n-1)RI}<0.1 \tag{3.1.1}$$

That is

$$CR=\frac{CI}{RI}<0.1\Leftrightarrow CI<0.1RI \tag{3.1.2}$$

$$\Leftrightarrow\frac{\lambda_{\max}-n}{n-1}<0.1RI \tag{3.1.3}$$

$$\Leftrightarrow\lambda_{\max}-n<0.1RI(n-1) \tag{3.1.4}$$

$$\Leftrightarrow\lambda_{\max}<0.1RI(n-1)+n \tag{3.1.5}$$

where the symbol '$\Leftrightarrow$' denotes equivalence. Let the right value be the maximum eigenvalue threshold $\lambda^n_{\max\ \text{threshold}}$ (in short $\lambda^n_{\max\ \text{thrd}}$), namely

$$\lambda^n{}_{\text{maxthrd}}=0.1RI(n-1)+n \tag{3.1.6}$$

Therefore:

$$CR<0.1\Leftrightarrow\lambda_{\max}<\lambda^n{}_{\max\ \text{thrd}} \tag{3.1.7}$$

$$\Leftrightarrow\nabla\lambda_{\max}=\lambda_{\max}-\lambda^{\text{n}}_{\max\ \text{thrd}}<0 \tag{3.1.8}$$

Since the CR method is mathematically equivalent to the maximum eigenvalue threshold method, the corresponding maximum eigenvalue threshold $\lambda^n_{\max\ \text{thrd}}$ of the comparison matrices with order n can be easily calculated using the formula (3.1.6) and the corresponding value of RI in Table 1. The results are listed in Table 2. □

Table 2
The threshold $\lambda^n_{\max\ \text{thrd}}$of the maximum eigenvalue and the corresponding RI.

n	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.52	0.89	1.11	1.25	1.35	1.4	1.45	1.49
$\lambda^n_{\max\ \text{thrd}}$	1	2	3.104	4.267	5.444	6.781	7.81	8.98	10.16	11.341

From Table 2, once the maximum eigenvalue λ_{max} of a comparison matrix is calculated, the consistency of this comparison matrix can be tested by comparing the $\lambda_{max}$with the maximum eigenvalue threshold $\lambda^{n}_{max\ thrd}$ without calculating the CI and CR. For instance, assume $\lambda_{max}=5.2$ for a comparison matrix of order 5, the comparison matrix is consistent because $\lambda_{max}=5.2<\lambda^{5}_{max\ thrd}=5.444$, which is equivalent to $CR=0.045<0.1$. Therefore, the maximum eigenvalue threshold $\lambda^{n}_{max\ thrd}$ can be used as a new consistency index for the ANP to test whether a comparison matrix is consistent. The specific principal of inconsistency test can be defined as follows:

Consistency test principal: If $\lambda^{i}_{max}<\lambda^{n}{}_{max\ \ thrd}$, that is, $\Delta\lambda^{i}_{max}<0$, then the *i*th comparison matrix passes the consistency test. If $\lambda^{i}_{max}\geq\lambda^{n}{}_{max\ \ threshold}$, that is, $\Delta\lambda^{i}_{max}\geq 0$, the *i*th comparison matrix fails the consistency test. The *i*th comparison matrix should be revised.

3.2. The advantage of the new consistency index for the ANP

The CR method is the most widely used consistency test method in the AHP/ANP. Although the CR method is mathematically equivalent to the maximum eigenvalue threshold $\lambda^{n}{}_{max\ \ threshold}$ method, that is, $CR=CI/RI<0.1$ is equivalent to $\lambda^{n}_{max}<\lambda^{n}{}_{max\ \ thrd}$ or $\Delta\lambda^{n}_{max}<0$, the maximum threshold $\lambda^{n}{}_{max\ \ threshold}$ method is easier to implement than the *CR* method.

The principals of consistency test of the *CR* method and the $\lambda^{n}_{maxthrd}$method are shown in inequalities (3.2.1) and (3.2.2), respectively:

$$CR=\frac{CI}{RI}=\frac{\lambda_{max}-n}{(n-1)RI}<0.1 \tag{3.2.1}$$

$$\lambda_{max}<\lambda^{n}_{max\ thrd}\ \text{or}\ \Delta\lambda^{n}_{max}<0 \tag{3.2.2}$$

The detailed processes of the *CR* method and the $\lambda^{n}_{max\ thrd}$method are shown as Figs. 3.1 and 3.2, respectively.

Compared to the CR method, in the $\lambda^{n}_{maxthrd}$method, there is no need to calculate the two middle steps, which saves $2N(n_1+n_2+\cdots+n_N)$ times in calculation. Therefore, the advantages of the $\lambda^{n}_{max\ thrd}$ method can be summarized into two aspects: efficient and easier to be implemented.

3.3. The block diagonal matrix for consistency test

In this section, the block diagonal matrix, based on the comparison matrices in the same level or different levels, is proposed to further simplify the processes of the consistency test in the ANP. As the AHP is a special case in the ANP, without losing generality, two typical cases of AHP structures with three levels and four levels, respectively, are used to illustrate the block diagonal matrix and the consistency test methodologies.

Case 1. The block diagonal matrix is constructed for a typical AHP model with three levels as shown in Fig. 3.3.

In this hierarchy structure, there are five comparison matrices: one with order four for the criteria with respect to the goal in the first level, denoted as *A*, and four with order three for the three alternatives with respect to the four criteria in the second level, denoted as *C*1, *C*2, *C*3, and *C*4. The consistency test for the comparison matrix *A* can be tested independently, while the other four comparison matrices with the same order can be tested simultaneously. The consistency test includes three steps:

Step 1: Construct the block diagonal matrix (*B* in short) using the five comparison matrices as the entries in the main diagonal:

$$B=\begin{pmatrix} A & & & & \\ & C1 & & & \\ & & C2 & & \\ & & & C3 & \\ & & & & C4 \end{pmatrix} \tag{3.3.1}$$

Step 2: Calculate the eigenvalues in the block diagonal matrix *B*. According to the notations of the block diagonal matrix, the maximum eigenvalues in the block diagonal matrix are the corresponding maximum eigenvalues of the comparison matrices *A*, *C*1, *C*2, *C*3, and *C*4 respectively, which are denoted as $\lambda^{i}{}_{max}$ ($i=0,1,2,3,4$).

Step 3: Calculate the maximum eigenvalue bias $\Delta\lambda^{i}_{max}(i=0,1,2,3,4)$ using the following formulas, and judge its consistency using the corresponding condition mentioned above:

$$\Delta\lambda^{0}_{max}=\lambda^{0}_{max}-\lambda^{4}_{maxthrd} \tag{3.3.2}$$

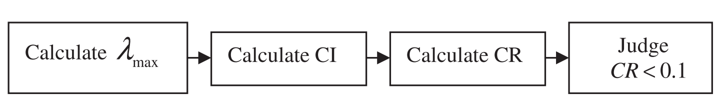

Fig. 3.1. The calculation processes of the CR method.

Fig. 3.2. The calculation processes of the $\lambda^{n}_{max\ thrd}$method.

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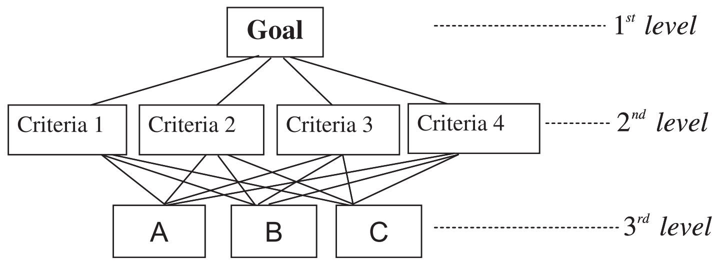

Fig. 3.3. The typical hierarchy structure with three levels in the AHP.

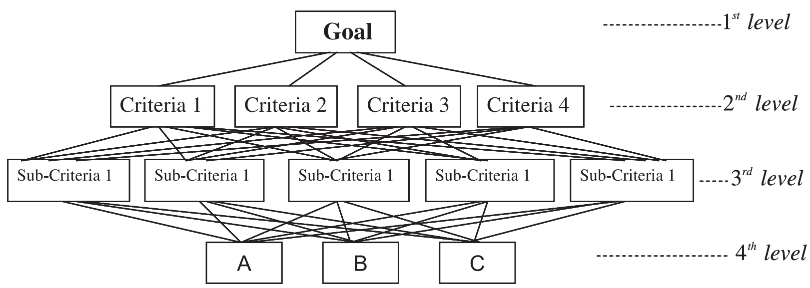

Fig. 3.4. The typical hierarchy structure with four levels in the AHP.

$$\Delta\lambda^i_{\max} = \begin{pmatrix} \lambda^1_{\max} & \lambda^2_{\max} & \lambda^3_{\max} & \lambda^4_{\max} \end{pmatrix} - \lambda^3_{\text{maxthrd}} \quad (3.3.3)$$

If $\Delta\lambda^i_{\max} < 0$ $(i=0,1,2,3,4)$, then the ith comparison matrix passes the consistency test, otherwise, it fails the consistency test. For instance, assume $\Delta\lambda^2_{\max} < 0$ and $\Delta\lambda^4_{\max} > 0$, the comparison matrix $C2$ passes the consistency test and is consistent while the comparison matrix $C4$ failed the consistency test, and its elements should be revised.

Case 2. The block diagonal matrix is constructed for a typical AHP structure with four levels as shown in Fig. 3.4.

In this four-level structure, there are ten pairwise comparison matrices.

In the 1st level: one matrix with order four with respect to the Goal, denoted as G.

In the 2nd level: four comparison matrices with order five for the five sub-criteria with respect to all four criteria, denoted as $C1$, $C2$, $C3$ and $C4$, respectively.

In the 3rd level: five comparison matrices for the three alternatives with respect to all the five sub-criteria, denoted as S1, S2, S3, S4, and S5, respectively.

After calculating each $\lambda^i_{\max}(i=0,1,2,\ldots,9)$ for the ten comparison matrices, one has to calculate the CR ten times for ten comparison matrices before judging whether the CR is less than 0.1. The complicities of CR calculation will be increased with the increase of the comparison matrices in the ANP. However, if the proposed maximum eigenvalue threshold index is used to test each consistency issue for ten comparison matrices, all the inconsistencies can be tested using the maximum eigenvalue threshold method, $\lambda^i_{\max} < \lambda^n_{\max\ \text{thrd}}$. For instance, comparing the $\lambda^0_{\max}$ with the threshold $\lambda^4{}_{\max\ \text{thrd}}$ for the first comparison matrix in order four, comparing the $\lambda^i_{\max}(i=1,\ldots,4)$ with the threshold $\lambda^5_{\max\ \text{thrd}}$ for the second four comparison matrices in order five, and comparing the $\lambda^i_{\max}(i=5,\ldots,9)$ with the threshold $\lambda^3{}_{\max\ \text{thrd}}$ for the last five comparison matrices in order three.

In the above case, there are two basic principles of consistency test using the maximum eigenvalues, as shown below:

Basic principle 1: level-by-level test—test the consistencies of the comparison matrices for each level. That is, test the consistencies of the comparison matrices with the same order in the same level one by one.

The processes of this method include the following steps:

Step 1: Construct the corresponding block diagonal matrix denoted as B_1, B_2, and B_3 for the comparison matrices in each level using the corresponding comparison matrices as the entries in the main diagonals:

$$B_1 = G \quad (3.3.4)$$

$$B_2 = \begin{pmatrix} C1 & & & \\ & C2 & & \\ & & C3 & \\ & & & C4 \end{pmatrix} \quad (3.3.5)$$

$$B_3 = \begin{pmatrix} S1 & & & & \\ & S2 & & & \\ & & S3 & & \\ & & & S4 & \\ & & & & S5 \end{pmatrix} \quad (3.3.6)$$

Note: If the order of the block diagonal matrix is acceptable, then the block diagonal matrix B can be constructed by all comparison matrices in the whole level as the entries in the main diagonal to calculate the eigenvalues of all comparison matrices simultaneously.

$$B=\begin{pmatrix} G &&&&&&&&& \\ & C1 &&&&&&&& \\ && C2 &&&&&&& \\ &&& C3 &&&&&& \\ &&&& C4 &&&&& \\ &&&&& S1 &&&& \\ &&&&&& S2 &&& \\ &&&&&&& S3 && \\ &&&&&&&& S4 & \\ &&&&&&&&& S5 \end{pmatrix} \tag{3.3.7}$$

Step 2: Calculate the eigenvalues in the block diagonal matrices B_1, B_2, and B_3 respectively. Or directly calculate the eigenvalues of the block diagonal matrix B. According to the notations of the block diagonal matrix, the maximum eigenvalues in the block diagonal matrix are the corresponding maximum eigenvalues of the comparison matrices G, $C1$, $C2$, $C3$, $C4$, $S1$, $S2$, $S3$, $S4$, and $S5$, respectively, denoted as $\lambda^{i}{}_{\max}$ (i=0,1,2,3,4,5,6,7,8,9).

Step 3: Calculate the maximum eigenvalue biases for the comparison matrices for each level using the following formulas, and judge its consistency using the corresponding condition mentioned above, that is, $\Delta\lambda^0_{\max}$ for the first level, $\Delta\lambda^i_{\max}$ (i=1,2,3,4) for the second level, and $\Delta\lambda^i_{\max}$(i=5,6,7,8,9) for the third level.

The 1st level:

$$\Delta\lambda^0_{\max}=\lambda^0_{\max}-\lambda^4_{\max\ \mathrm{thrd}} \tag{3.3.8}$$

The 2nd level:

$$\Delta\lambda^i_{\max}=\begin{pmatrix}\lambda^1_{\max} & \lambda^2_{\max} & \lambda^3_{\max} & \lambda^4_{\max}\end{pmatrix}-\lambda^5_{\max\ \mathrm{thrd}} \tag{3.3.9}$$

The 3rd level:

$$\Delta\lambda^i_{\max}=\begin{pmatrix}\lambda^5_{\max} & \lambda^6_{\max} & \lambda^7_{\max} & \lambda^8_{\max} & \lambda^9_{\max}\end{pmatrix}-\lambda^3_{\max\ \mathrm{thrd}} \tag{3.3.10}$$

If $\Delta\lambda^i_{\max}<0$ (i=0,1,2,3,4,5,6,7,8,9), then the ith comparison matrix passes the consistency test, otherwise, it fails the consistency test. The specific processes of consistency test are the same processes illustrated previously.

Basic Principle 2: Whole-level test—test the whole consistencies of the comparison matrices simultaneously.

The processes of this method include:

Step 1: Construct the block diagonal matrix B using all the comparison matrices as the entries in the main diagonals at one time (simultaneously):

$$B=\begin{pmatrix} G &&&&&&&&& \\ & C1 &&&&&&&& \\ && C2 &&&&&&& \\ &&& C3 &&&&&& \\ &&&& C4 &&&&& \\ &&&&& S1 &&&& \\ &&&&&& S2 &&& \\ &&&&&&& S3 && \\ &&&&&&&& S4 & \\ &&&&&&&&& S5 \end{pmatrix} \tag{3.3.11}$$

Step 2: Calculate the eigenvalues of the block diagonal matrix B. Then the maximum eigenvalues of the comparison matrices G, $C1$, $C2$, $C3$, $C4$, $S1$, $S2$, $S3$, $S4$, and $S5$ can be calculated and denoted as $\lambda^{i}{}_{\max}$ (i=0,1,2,3,4,5,6,7,8,9), respectively.

Step 3: Calculate the maximum eigenvalue biases for all comparison matrices using the following formula, and judge its consistency using the corresponding condition mentioned above:

$$\begin{aligned}\Delta\lambda^i_{\max}=&(\lambda^0_{\max}\quad \lambda^1_{\max}\quad \lambda^2_{\max}\quad \lambda^3_{\max}\quad \lambda^4_{\max}\quad \lambda^5_{\max}\quad \lambda^6_{\max}\quad \lambda^7_{\max}\quad \lambda^8_{\max}\quad \lambda^9_{\max})\\ &-(\lambda^4_{\mathrm{maxthrd}}\quad \lambda^5_{\mathrm{maxthrd}}\quad \lambda^5_{\mathrm{maxthrd}}\quad \lambda^5_{\mathrm{maxthrd}}\quad \lambda^5_{\mathrm{maxthrd}}\quad \lambda^5_{\mathrm{maxthrd}}\quad \lambda^3_{\mathrm{maxthrd}}\quad \lambda^3_{\mathrm{maxthrd}}\quad \lambda^3_{\mathrm{maxthrd}}\quad \lambda^3_{\max\ \mathrm{thrd}})\end{aligned} \tag{3.3.12}$$

Likewise, if $\Delta\lambda^i_{\max}<0$ (i=0,1,2,3,4,5,6,7,8,9), then the ith comparison matrix passes the consistency test, otherwise, it fails the consistency test. The specific processes of consistency test are the same processes illustrated previously.

3.4. The induced bias block diagonal matrix and inconsistency identification methodologies

The consistency test method for a reciprocal pairwise comparison matrix (RPCM) has been proposed in previous sections. In this section, an induced bias block diagonal matrix is introduced to identify the inconsistent elements in the reciprocal pairwise comparison matrix. To present the inconsistency identification method, three definitions are introduced below.

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Definition 1. A comparison matrix A is positive reciprocal matrix if $a_{ii}=1$, $a_{ij}>0$ and $a_{ij}=1/a_{ji}$ for all positive integer i and j.

Definition 2. A reciprocal matrix is perfectly consistent if $a_{ik}a_{kj}=a_{ij}$ for all i, j, and k.

Definition 3. A reciprocal matrix is approximately consistent if $a_{ik}a_{kj}\approx a_{ij}$ for all i, j, and k, where '$\approx$' denotes approximately or close to.

According to the three definitions for reciprocal matrix, the theorems of matrix multiplication, and vectors dot product, the following statements can be derived [9]:

Theorem 1. *The induced bias matrix $C=AA-nA$ should be a zero matrix if comparison matrix A is perfectly consistent, where n is the size of the matrix A.*

Corollary 1. *The induced bias matrix $C=AA-nA$ should be as close as possible to zero matrix if comparison matrixAis approximately consistent.*

Corollary 2. *There must be some inconsistent elements in induced bias matrix C deviating far away from zero if the pairwise matrix is inconsistent.*

The above theorem and corollaries are proposed for a single reciprocal pairwise comparison matrix, according to the properties of a block diagonal matrix and the Theorem 1 and the two corollaries, the following theorem and corollaries can be derived:

Theorem 1′. *The induced bias block matrix $C=BB-nB$ should be a zero matrix if the sub-matrices along the main diagonal of the block diagonal comparison matrix B are perfectly consistent, where the block diagonal comparison matrix B is composed of the comparison matrices within the same or different level, n denotes the sizes of the comparison matrices, if the sizes of the sub-matrices in the block diagonal matrix are identical; n denotes any of the size of the sub-matrix, if the sizes of the sub-matrices are different; and n denotes a diagonal matrix consisting of the sizes of all the sub-matrices along the main diagonal, which is denoted as $n=diag(n_1,n_2,...,n_n)$.*

Corollary 1′. *The induced bias block diagonal matrix $C=BB-nB$ should be as close as possible to zero matrix if the sub-matrices of the block diagonal comparison matrix B are approximately consistent.*

Corollary 2′. *There must be some inconsistent elements in the corresponding induced bias sub-matrices of the induced bias block diagonal matrix C deviating far away from zero if the corresponding sub-matrices along the main diagonal are inconsistent.*

The identification principals include two approaches for the different levels.

Gradual-level identification principal: construct the induced bias block matrix $C_i=B_iB_i-nB_i$ for each level using the inconsistent block diagonal matrix to identify the inconsistent elements, where B_i is the ith inconsistent block diagonal matrix consisting of the inconsistent matrices in the ith level and n is the dimension of each inconsistent matrix in the same level. Usually the sizes of the matrices are identical in each level. If the sizes of the matrices are unequal, the whole-level identification principal should be applied, otherwise, the specific processes of identifying the inconsistent elements include the following 7 steps:

Step 1: Construct the inconsistent block diagonal matrix for each level using the inconsistent comparison matrices. Take Fig. 3.4 as an example; assume there exist the following inconsistent matrices: matrix G in the first level, matrices $C1$, $C2$, and $C3$ in the second level, and matrices $S1$, $S2$, $S3$, and $S4$ in the third level, then we can construct three block diagonal matrices as shown below.

The 1st level:

$$B_1=G \tag{3.4.1}$$

The 2nd level:

$$B_2=\begin{pmatrix} C1 & & \\ & C2 & \\ & & C3 \end{pmatrix} \tag{3.4.2}$$

The 3rd level:

$$B_3=\begin{pmatrix} S1 & & & \\ & S2 & & \\ & & S3 & \\ & & & S4 \end{pmatrix} \tag{3.4.3}$$

Step 2: Introduce the induced bias block diagonal matrices for each level to identify the farthest value from zero in each sub-matrix along the main diagonal.

The 1st level:

$$C_1=B_1B_1-4B_1=GG-4G \tag{3.4.4}$$

The 2nd level:

$$C_2=B_2B_2-5B_2=\begin{pmatrix} C1 & & \\ & C2 & \\ & & C3 \end{pmatrix}\begin{pmatrix} C1 & & \\ & C2 & \\ & & C3 \end{pmatrix}-5\begin{pmatrix} C1 & & \\ & C2 & \\ & & C3 \end{pmatrix}=\begin{pmatrix} C1\times C1-5\times C1 & & \\ & C2\times C2-5\times C2 & \\ & & C3\times C3-5\times C3 \end{pmatrix} \tag{3.4.5}$$

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The 3rd level:

$$C_3 = B_3B_3 - 3B_3 = \begin{pmatrix} S1 & & & \\ & S2 & & \\ & & S3 & \\ & & & S4 \end{pmatrix}\begin{pmatrix} S1 & & & \\ & S2 & & \\ & & S3 & \\ & & & S4 \end{pmatrix} - 3\begin{pmatrix} S1 & & & \\ & S2 & & \\ & & S3 & \\ & & & S4 \end{pmatrix} = \begin{pmatrix} S1\times S1-3\times S1 & & & \\ & S2\times S2-3\times S2 & & \\ & & S3\times S3-3\times S3 & \\ & & & S4\times S4-3\times S4 \end{pmatrix} \quad (3.4.6)$$

Step 3: Identify the largest absolute values of elements in each induced bias sub-matrix of the induced bias block matrix C_i. Take the induced bias block diagonal matrix C_2 in the 2nd level as an example; suppose the largest absolute values in the three induced bias sub-matrices $C1\times C1-5\times C1$, $C2\times C2-5\times C2$, and $C3\times C3-5\times C3$ are $c_{ij}(1\le i,j\le 5$, and $i\ne j)$. These largest absolute values correspond to the elements c_{ij}, $c_{5+i,5+j}$ and $c_{10+i,10+j}$ in the bias block comparison matrix B, respectively, which are located at ith row and jth column, $(5+i)$th row and $(5+j)$th column, and $(10+i)$th row and $(10+j)$th column, respectively.

Step 4: Construct the following row vectors and column vectors with the corresponding elements of the identified rows and columns in Step 3:

$$r_i = (b_{i1},b_{i2},\ldots,b_{i,15}) = (b_{i1},b_{i2},b_{i3},b_{i4},b_{i5},0,\ldots,0) \quad \text{and} \quad c_j^T = (b_{1j},b_{2j},\ldots,b_{15,j})^T = (b_{1j},b_{2j},b_{3j},b_{4j},b_{5j},0,\ldots,0)^T \quad (3.4.7)$$

$$r_{5+i} = (b_{5+i,1},b_{5+i,2},\ldots,b_{5+i,15}) = (0,\ldots,0,b_{5+i,6},b_{5+i,7},b_{5+i,8},b_{5+i,9},b_{5+i,10},0,\ldots,0)$$
and
$$c_{5+j}^T = (b_{1,5+j},b_{2,5+j},\ldots,b_{15,5+j})^T = (0,\ldots,0,b_{6,5+j},b_{7,5+j},b_{8,5+j},b_{9,5+j},b_{10,5+j},0,\ldots,0)^T \quad (3.4.8)$$

$$r_{10+i} = (b_{10+i,1},b_{10+i,2},\ldots,b_{10+i,15}) = (0,\ldots,0,b_{10+i,11},b_{10+i,12},b_{10+i,13},b_{10+i,14},b_{10+i,15})$$
and
$$c_{10+j}^T = (b_{1,10+j},b_{2,10+j},\ldots,b_{15,10+j})^T = (0,\ldots,0,b_{11,10+j},b_{12,10+j},b_{13,10+j},b_{14,10+j},b_{15,10+j})^T \quad (3.4.9)$$

For simplicity, one can remove the zeroes out of the vectors.

Step 5: Calculate the scalar products of the corresponding row and column vectors in n dimension. The dot products b_i $(i=1,2,3)$ become

$$b_1 = r_i\cdot c_j^T = (b_{i1}b_{1j},b_{i2}b_{2j},b_{i3}b_{3j},b_{i4}b_{4j},b_{i5}b_{5j},0,\ldots,0) \quad (3.4.10)$$

$$b_2 = r_{5+i}\cdot c_{5+j}^T = (0,\ldots,0,b_{5+i,6}b_{6,5+j},b_{5+i,7}b_{7,5+j},b_{5+i,8}b_{8,5+j},b_{5+i,9}b_{9,5+j},b_{5+i,10}b_{10,5+j},0,\ldots,0) \quad (3.4.11)$$

$$b_3 = r_{10+i}\cdot c_{10+j}^T = (0,\ldots,0,b_{10+i,11}b_{11,10+j},b_{10+i,12}b_{12,10+j},b_{10+i,13}b_{13,10+j},b_{10+i,14}b_{14,10+j},b_{10+i,15}b_{15,10+j}) \quad (3.4.12)$$

Step 6: Compute the deviation elements that are far away from the elements b_{ij}, $b_{5+i,5+j}$, and $b_{10+i,10+j}$ in the vectors b_i $(i=1,2,3)$ using the following formulas. Let f_i be the bias identifying vectors henceforth:

$$f_1 = b_1 - b_{ij} = (b_{i1}b_{1j}-b_{ij},b_{i2}b_{2j}-b_{ij},b_{i3}b_{3j}-b_{ij},b_{i4}b_{4j}-b_{ij},b_{i5}b_{5j}-b_{ij},-b_{ij},\ldots,-b_{ij}) \quad (3.4.13)$$

$$f_2 = b_2 - b_{5+i,5+j} = (-b_{5+i,5+j},\ldots,-b_{5+i,5+j},b_{5+i,6}b_{6,5+j}-b_{5+i,5+j},\ldots,b_{5+i,10}b_{10,5+j}-b_{5+i,5+j},-b_{5+i,5+j},\ldots,-b_{5+i,5+j}) \quad (3.4.14)$$

$$f_3 = b_3 - b_{10+i,10+j} = (-b_{10+i,10+j},\ldots,-b_{10+i,10+j},b_{10+i,11}b_{11,10+j}-b_{10+i,10+j},\ldots,b_{10+i,15}b_{15,10+j}-b_{10+i,10+j}) \quad (3.4.15)$$

Step 7: Identify the possible inconsistent elements that have the largest absolute values in the scalar products b_i $(i=1,2,3)$ and the bias identifying vectors f_i $(i=1,2,3)$. Suppose $b_{i2}b_{2j}$ in b_1, $b_{5+i,7}b_{7,5+j}$ in b_2, and $b_{10+i,11}b_{11,10+j}$ in b_3 are the elements with the largest absolute values. The adjustment direction of these inconsistent elements can be identified by observing the corresponding values in the bias block comparison matrix C as shown below:

If $c_{i2}<0 \Rightarrow b_{i2}\uparrow \Leftrightarrow c1_{i2}\uparrow$ vice versa, $c1_{i2}\downarrow$
If $c_{2j}<0 \Rightarrow b_{2j}\uparrow \Leftrightarrow c1_{2j}\uparrow$ vice versa, $c1_{2j}\downarrow$
If $c_{5+i,7}<0 \Rightarrow b_{5+i,7}\uparrow \Leftrightarrow c2_{i2}\uparrow$ vice versa, $c2_{i2}\downarrow$
If $c_{7,5+j}<0 \Rightarrow b_{7,5+j}\uparrow \Leftrightarrow c2_{2j}\uparrow$ vice versa, $c2_{2j}\downarrow$
If $c_{10+i,11}<0 \Rightarrow b_{10+i,11}\uparrow \Leftrightarrow c3_{i1}\uparrow$ vice versa, $c3_{i1}\downarrow$
If $c_{11,10+j}<0 \Rightarrow b_{11,10+j}\uparrow \Leftrightarrow c3_{1j}\uparrow$ vice versa, $c3_{1j}\downarrow$

where the symbol $\uparrow$ denotes the element is too large while $\downarrow$ denotes the element is too small. Therefore, the inconsistent elements in the comparison matrices $C1$, $C2$, and $C3$ are identified, and need to be revised. The product of the corresponding revised values should be equal or close to the deviating values of elements. For instance

$$b_{i2}b_{2j} = b_{ij} \text{ or} \approx b_{ij}$$

$$b_{5+i,7}b_{7,5+j} = b_{5+i,5+j} \text{ or} \approx b_{5+i,5+j}$$

$$b_{10+i,11}b_{11,10+j} = b_{10+i,10+j} \text{ or} \approx b_{10+i,10+j}$$

Whole-level identification principal: Construct the whole induced bias block matrix $C=BB-nB$ using all the inconsistent block diagonal matrix to identify the inconsistent elements, where B is the whole inconsistent block diagonal matrix consisting of all the inconsistent matrices along the main diagonal and n is the matrix sizes denoted as a dimension vector $n=\text{diag}(n_1,n_2,\ldots,n_n)$, where n_i represents a block diagonal matrix of the corresponding dimension of each inconsistent matrix.

The processes are as follows:
Step 1: Construct the whole block matrix using all the inconsistent matrices:

$$B=\begin{pmatrix} G &&&&&&& \\ & C1 &&&&&& \\ && C2 &&&&& \\ &&& C3 &&&& \\ &&&& S1 &&& \\ &&&&& S2 && \\ &&&&&& S3 & \\ &&&&&&& S4 \end{pmatrix} \tag{3.4.16}$$

Step 2: Introduce the whole induced bias block matrix using the formula $C=BB-nB$. Then

$$C=BB-nB=\begin{pmatrix} G &&&&&&& \\ & C1 &&&&&& \\ && C2 &&&&& \\ &&& C3 &&&& \\ &&&& S1 &&& \\ &&&&& S2 && \\ &&&&&& S3 & \\ &&&&&&& S4 \end{pmatrix}\begin{pmatrix} G &&&&&&& \\ & C1 &&&&&& \\ && C2 &&&&& \\ &&& C3 &&&& \\ &&&& S1 &&& \\ &&&&& S2 && \\ &&&&&& S3 & \\ &&&&&&& S4 \end{pmatrix}$$

$$-\begin{pmatrix} 4 &&&&&&& \\ & 5 &&&&&& \\ && 5 &&&&& \\ &&& 5 &&&& \\ &&&& 3 &&& \\ &&&&& 3 && \\ &&&&&& 3 & \\ &&&&&&& 3 \end{pmatrix}\begin{pmatrix} G &&&&&&& \\ & C1 &&&&&& \\ && C2 &&&&& \\ &&& C3 &&&& \\ &&&& S1 &&& \\ &&&&& S2 && \\ &&&&&& S3 & \\ &&&&&&& S4 \end{pmatrix}$$

$$=\begin{pmatrix} G\times G-4\times G &&&&&&& \\ & C1\times C1-5\times C1 &&&&&& \\ && C2\times C2-5\times C2 &&&&& \\ &&& C3\times C3-5\times C3 &&&& \\ &&&& S1\times S1-3\times S1 &&& \\ &&&&& S2\times S2-3\times S2 && \\ &&&&&& S3\times S3-3\times S3 & \\ &&&&&&& S4\times S4-3\times S4 \end{pmatrix} \tag{3.4.17}$$

The processes of identifying the inconsistent elements of the sub-matrices along the main diagonal are the same as gradual-level test method.

4. Illustrative examples

To further illustrate the maximum eigenvalue threshold index and the induced bias block matrix for rapidly identifying the inconsistent elements of the comparison matrices when the ANP is applied in risk assessment, decision analysis, and emergency management, two public-domain examples are revised to test the proposed method.

Example 1. With the development of the information communication technology (ICT), the computer system (CS) is increasingly important and becoming one of the central components in risk assessment and decision analysis. However, CS is not reliable in emergent events such as earthquake, fire emergence, or even stolen/hi-jack. In such scenarios, the emergency managers have to select the most reliable computer system. For simplicity, an example selecting the best computer system first introduced by Triantaphyllou and Mann [49],

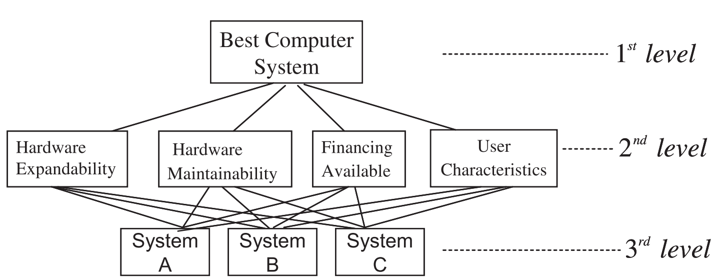

Fig. 4.1. The structure of selecting the best computer system.

which has the typical hierarchy structure as shown in Fig. 4.1, is used to test the proposed method. Suppose there are three alternative configurations, system A, system B, and system C. There are also four criteria: hardware expandability, hardware maintainability, financing available, and user friendly characteristics of the operating system and related available software, denoted as C1, C2, C3, and C4 respectively.

Since the gradual-level consistency test and inconsistency identification method is relatively simple compared with the whole-level method, therefore, the whole level method is used to test the consistency and identify the inconsistent elements in this example. The whole process can be divided into two stages: consistency test and inconsistency identification.

Stage I: Test the consistency of all the comparison matrices using whole-level method

Step 1: Construct the block diagonal matrix *B* showed below using the matrices *A*, *C*1, *C*2, *C*3, and *C*4:

Columns 1 through 11 Columns 12 through 16

1.0000	5.0000	3.0000	7.0000	0	0	0	0	0	0	0	0	0	0	0	0
0.2000	1.0000	0.3333	5.0000	0	0	0	0	0	0	0	0	0	0	0	0
0.3333	3.0000	1.0000	6.0000	0	0	0	0	0	0	0	0	0	0	0	0
0.1429	0.2000	0.1667	1.0000	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1.0000	6.0000	8.0000	0	0	0	0	0	0	0	0	0
0	0	0	0	0.1667	1.0000	4.0000	0	0	0	0	0	0	0	0	0
0	0	0	0	0.1250	0.2500	1.0000	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1.0000	7.0000	0.2000	0	0	0	0	0	0
0	0	0	0	0	0	0	0.1429	1.0000	0.1250	0	0	0	0	0	0
0	0	0	0	0	0	0	5.0000	8.0000	1.0000	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1.0000	8.0000	6.0000	0	0	0
0	0	0	0	0	0	0	0	0	0	0.1250	1.0000	0.2500	0	0	0
0	0	0	0	0	0	0	0	0	0	0.1667	4.0000	1.0000	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1.0000	5.0000	4.0000
0	0	0	0	0	0	0	0	0	0	0	0	0	0.2000	1.0000	0.3333
0	0	0	0	0	0	0	0	0	0	0	0	0	0.2500	3.0000	1.0000

Step 2: Calculate the eigenvalue of block diagonal matrix *B*, and identify the maximum eigenvalues of the corresponding block diagonal sub-matrix *A*, *C*1, *C*2, *C*3, and *C*4. That is

$$\lambda^0_{\max}=4.2365,\ \lambda^1_{\max}=3.1356,\ \lambda^2_{\max}=3.2470,\ \lambda^3_{\max}=3.1356,\ \lambda^4_{\max}=3.0858$$

Step 3: Test the consistency using the maximum eigenvalue threshold method. That is

$$\Delta\lambda^i_{\max}=\left(\lambda^0_{\max},\lambda^1_{\max},\ \lambda^2_{\max},\ \lambda^3_{\max},\ \lambda^4_{\max}\right)-\left(\lambda^4_{\mathrm{maxthrd}},\lambda^3_{\mathrm{maxthrd}},\ \lambda^3_{\mathrm{maxthrd}},\lambda^3_{\mathrm{maxthrd}},\lambda^3_{\mathrm{maxthrd}}\right)=(3.1356,\ 3.2470,\ 3.1356,\ 3.0858)-(4.267,\ 3.104,\ 3.104,\ 3.104,\ 3.104)$$

$$=(-0.0305,\ 0.0316,\ 0.143,\ 0.0316,\ -0.0182)$$

Obviously, only$\Delta\lambda^0_{\max}<0$ and $\Delta\lambda^4_{\max}<0$, which means only the two comparison matrices *A* and *C*4 are consistent, and other matrices are inconsistent.

Stage II: Identify the inconsistent elements using the whole level identification principal

Step 1: Construct the whole block matrix *B* using all the inconsistent matrices *C*1, *C*2, and *C*3.

1.0000	6.0000	8.0000	0	0	0	0	0	0
0.1667	1.0000	4.0000	0	0	0	0	0	0
0.1250	0.2500	1.0000	0	0	0	0	0	0
0	0	0	1.0000	7.0000	0.2000	0	0	0
0	0	0	0.1429	1.0000	0.1250	0	0	0
0	0	0	5.0000	8.0000	1.0000	0	0	0
0	0	0	0	0	0	1.0000	8.0000	6.0000
0	0	0	0	0	0	0.1250	1.0000	0.2500
0	0	0	0	0	0	0.1667	4.0000	1.0000

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Step 2: Introduce the whole induced bias block matrix using the formula $C = BB - 3B$. That is

0	−4.0000	16.0000	0	0	0	0	0	0
0.3333	0	−2.6667	0	0	0	0	0	0
−0.0833	0.5000	0	0	0	0	0	0	0
0	0	0	0	−5.4000	0.6750	0	0	0
0	0	0	0.4821	0	−0.0964	0	0	0
0	0	0	−3.8571	27.0000	0	0	0	0
0	0	0	0	0	0	0	16.0000	−4.0000
0	0	0	0	0	0	−0.0833	0	0.5000
0	0	0	0	0	0	0.3333	−2.6667	0

Step 3: The largest values, 16, 27, and 16, in each sub-matrix, are located at 1st row and 3rd column, 6th row and 5th column, and 7th row and 8th column, respectively.

Step 4: The corresponding vectors are

$$r_1 = \begin{pmatrix} 1 & 6 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$c_3^T = \begin{pmatrix} 8 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$r_6 = \begin{pmatrix} 0 & 0 & 0 & 5 & 8 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$c_5^T = \begin{pmatrix} 0 & 0 & 0 & 7 & 1 & 8 & 0 & 0 & 0 \end{pmatrix}$$

$$r_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & 6 \end{pmatrix} \quad \text{and}$$

$$c_8^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 8 & 1 & 4 \end{pmatrix}$$

Step 5: The corresponding scalar products b_i $(i=1,2,3)$ are

$$b_1 = r_1 \cdot c_3^T = \begin{pmatrix} 8 & 24 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$b_2 = r_6 \cdot c_5^T = \begin{pmatrix} 0 & 0 & 0 & 35 & 8 & 8 & 0 & 0 & 0 \end{pmatrix}$$

$$b_3 = r_7 \cdot c_8^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 24 \end{pmatrix}$$

Step 6: The corresponding bias identifying vectors f_i $(i=1,2,3)$ are

$$f_1 = r_1 \cdot c_3^T - b_{13} = \begin{pmatrix} 0 & 16 & 0 & -8 & -8 & -8 & -8 & -8 & -8 \end{pmatrix}$$

$$f_2 = r_6 \cdot c_5^T - b_{65} = \begin{pmatrix} -8 & -8 & -8 & 27 & 0 & 0 & -8 & -8 & -8 \end{pmatrix}$$

$$f_3 = r_7 \cdot c_8^T - b_{78} = \begin{pmatrix} -8 & -8 & -8 & -8 & -8 & -8 & 0 & 0 & 16 \end{pmatrix}$$

Step 7: From the scalar products b_i $(i=1,2,3)$ and the bias identifying vectors f_i($i=1,2,3$), the inconsistent values are $b_{12}b_{23}=24$, $b_{64}b_{45}=35$, and $b_{79}b_{98}=24$, which are correspondingly the inconsistent elements $C1_{12}C1_{23}=24$, $C2_{31}C2_{12}=35$, and $C3_{13}C3_{32}=24$.

Furthermore, $c_{12}=-4<0$, $c_{23}=-2.6667<0$, $c_{64}=-3.8571<0$, $c_{45}=-5.4<0$, $c_{79}=-4<0$, and $c_{98}=-2.6667<0$. These inequalities indicate that all values of b_{12}, b_{23}, b_{64}, b_{45}, b_{79}, and b_{98} are probably too large. That is, $C1_{12}$, $C1_{23}$, $C2_{31}$, $C2_{12}$, $C3_{13}$, and $C3_{32}$ are too large, and their values should be decreased. Namely, $C1_{12}C1_{23}=8$, $C2_{31}C2_{12}=8$, and $C3_{13}C3_{32}=8$. For instance:

$$C1_{12}=4,\ C1_{21}=\tfrac{1}{4},\ C1_{23}=2,\ C1_{32}=\tfrac{1}{2}$$

$$C2_{31}=2,\ C2_{13}=\tfrac{1}{2},\ C2_{12}=4,\ C2_{21}=\tfrac{1}{4}$$

$$C3_{13}=4,\ C3_{31}=\tfrac{1}{4},\ C3_{32}=2,\ C3_{23}=\tfrac{1}{2}$$

Then, the corresponding values in the comparison matrices $C1$, $C2$, and $C3$ are replaced with the above values, and reconstruct the block diagonal matrix B, which is shown as follows:

1.0000	4.0000	8.0000	0	0	0	0	0	0
0.2500	1.0000	2.0000	0	0	0	0	0	0
0.1250	0.5000	1.0000	0	0	0	0	0	0
0	0	0	1.0000	4.0000	0.5000	0	0	0
0	0	0	0.2500	1.0000	0.1250	0	0	0
0	0	0	2.0000	8.0000	1.0000	0	0	0
0	0	0	0	0	0	1.0000	8.0000	4.0000
0	0	0	0	0	0	0.1250	1.0000	0.5000
0	0	0	0	0	0	0.2500	2.0000	1.0000

The corresponding maximum values of the diagonal sub-matrices in the block diagonal matrix B can be calculated by calculating the eigenvalue of matrix B. Three of the maximum values are 3, which are equal to the order 3, therefore, they are consistent comparison matrices.

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We also can calculate the bias matrix C using the formula $C=BB-3B$, which is listed below:

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Therefore, the consistencies of all the comparison matrices can easily be tested simultaneously by the proposed maximum eigenvalue index, and the inconsistent elements of the inconsistent comparison matrices can also be identified and adjusted simultaneously by the proposed whole-level identification method.

Example 2. In Wenchuan Earthquake, many schools were destroyed. Survived students, especially high middle school students, have to transfer and select another school to continue their study. The ANP can be used to help the students selecting the best school. To further illustrate the proposed method in such scenario, the best school selection example first introduced by Saaty [37] is used in this paper. In addition, this example was again used by Saaty [46] to illustrate the ANP Formulation of the Classic AHP School Example. The school choice hierarchy is showed in Fig. 4.2. There are also six pairwise comparison matrices in this hierarchy structure. One with order six (6×6) with respect to the Goal, satisfaction with school, six comparison matrices with order three (3×3) with respect to the six criteria, Learning, Friends, School Life, Vocational Training, College Preparation, and Music Classes, which are connected to the three alternatives.

The whole-level consistency test and inconsistency identification methods have been illustrated in the previous example, therefore, the gradual-level method is used in this example to test the consistency and identify the inconsistent elements.

Stage I: Test the consistencies of all the comparison matrices using gradual-level method

Step 1: Construct the corresponding block diagonal matrices B_1 and B_2 showed below, respectively, using the matrix A in the first level, and the matrices $C1$, $C2$, $C3$, $C4$, $C5$, and $C6$ in the second level.

1.0000	4.0000	3.0000	1.0000	3.0000	4.0000
0.2500	1.0000	7.0000	3.0000	0.2000	1.0000
0.3333	0.1429	1.0000	0.2000	0.2000	0.1667
1.0000	0.3333	5.0000	1.0000	1.0000	0.3333
0.3333	5.0000	5.0000	1.0000	1.0000	3.0000
0.2500	1.0000	6.0000	3.0000	0.3333	1.0000

and

Columns 1 through 11 Columns 12 through 18

1.0000	0.3333	0.5000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3.0000	1.0000	3.0000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2.0000	0.3333	1.0000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1.0000	1.0000	1.0000	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1.0000	1.0000	1.0000	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1.0000	1.0000	1.0000	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1.0000	5.0000	1.0000	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0.2000	1.0000	0.2000	0.	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1.0000	5.0000	1.0000	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1.0000	9.0000	7.0000	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0.1111	1.0000	0.2000	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0.1429	5.0000	1.0000	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1.0000	0.50000	1.0000	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	2.0000	1.0000	2.0000	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1.0000	0.50000	1.0000	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1.0000	6.0000	4.0000
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.1667	1.0000	0.3333
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.2500	3.0000	1.0000

Step 2: Calculate the maximum eigenvalues of block diagonal matrices B_1 and B_2, and identify the corresponding eigenvalues of comparison matrices A, $C1$, $C2$, $C3$, $C4$, $C5$, and $C6$, denoted as $\lambda^i_{\max}(i=0,1,2,\ldots,6)$. We get

$$\lambda^0_{\max}=7.4199 \text{ and } \lambda^1_{\max}=3.0536,\ \lambda^2_{\max}=3,\ \lambda^3_{\max}=3,\ \lambda^4_{\max}=3.2085,\ \lambda^5_{\max}=3,\ \lambda^6_{\max}=3.0536$$

Step 3: Test the consistencies using the maximum eigenvalue threshold method. That is

$$\Delta\lambda^0_{\max}=\lambda^0{}_{\max}-\lambda^6{}_{\max\ \mathrm{thrd}}=7.4199-6.781=0.6389>0$$

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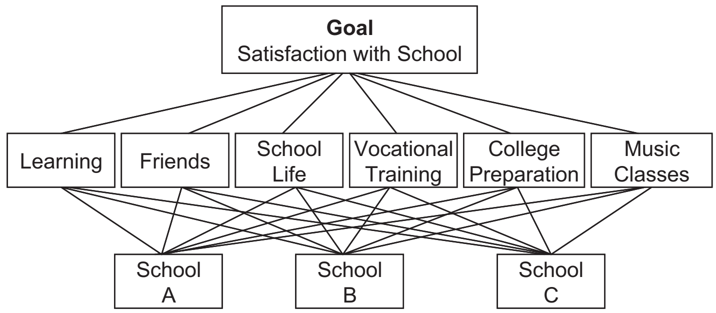

Fig. 4.2. The school choice hierarchy.

$$\Delta\lambda^{i}_{\max} = (\lambda^{1}_{\max},\ \lambda^{2}_{\max},\ \lambda^{3}_{\max},\ \lambda^{4}_{\max},\ \lambda^{5}_{\max},\ \lambda^{6}_{\max}) - \lambda^{3}_{\max\ \text{thrd}}$$
$$= (3.0536,\ 3,\ 3,\ 3.2085,\ 3,\ 3.0536) - 3.104 = (-0.0504,\ -0.1040,\ -0.1040,\ 0.1045,\ -0.1040,\ -0.0504)$$

Since $\Delta\lambda^{0}_{\max} = 0.6389 > 0$ and $\Delta\lambda^{4} = 0.1045 > 0$, therefore, the comparison matrices A and $C4$ are inconsistent. Go to the second stage. The whole-level identification method is used here, because there are only two inconsistent comparison matrices.

Stage II: Identify the inconsistent elements using the whole-level identification principal

Step 1: Construct the whole block matrix B using the inconsistent matrices A and $C4$.

$$\begin{matrix}
1.0000 & 4.0000 & 3.0000 & 1.0000 & 3.0000 & 4.0000 & 0 & 0 & 0 \\
0.2500 & 1.0000 & 7.0000 & 3.0000 & 0.2000 & 1.0000 & 0 & 0 & 0 \\
0.3333 & 0.1429 & 1.0000 & 0.2000 & 0.2000 & 0.1667 & 0 & 0 & 0 \\
1.0000 & 0.3333 & 5.0000 & 1.0000 & 1.0000 & 0.3333 & 0 & 0 & 0 \\
0.3333 & 5.0000 & 5.0000 & 1.0000 & 1.0000 & 3.0000 & 0 & 0 & 0 \\
0.2500 & 1.0000 & 6.0000 & 3.0000 & 0.3333 & 1.0000 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 9.0000 & 7.0000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.1111 & 1.0000 & 0.2000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.1429 & 5.0000 & 1.0000
\end{matrix}$$

Step 2: Introduce the whole induced bias block matrix using the formula:

$$C = BB - nB = \text{diag}(A,C4)\text{diag}(A,C4) - \text{diag}(6,3)\text{diag}(A,C4)$$

$$\begin{matrix}
0 & 3.7619 & 60.0000 & 23.6000 & -8.2667 & -2.1667 & 0 & 0 & 0 \\
4.6500 & 0 & -5.2500 & -7.1500 & 4.6833 & -0.2333 & 0 & 0 & 0 \\
-0.9893 & 1.9952 & 0 & 0.6619 & 0.4841 & 1.4762 & 0 & 0 & 0 \\
-1.8333 & 8.7143 & -7.6667 & 0 & 0.1778 & 6.8333 & 0 & 0 & 0 \\
3.3333 & -14.6190 & 39.0000 & 21.3333 & 0 & -4.5000 & 0 & 0 & 0 \\
4.3611 & 0.5238 & 0.4167 & -7.2167 & 3.8167 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 26.0000 & -5.2000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0825 & 0 & 0.5778 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.4127 & -3.7143 & 0
\end{matrix}$$

Step 3: The elements with largest absolute values in the sub-matrices A and $C4$ are 60 and 26 respectively, which are located at 1st row and 3rd column, 7th row and 8th column, respectively.

Step 4: The vectors are

$r_1 = (1\ \ 4\ \ 3\ \ 1\ \ 3\ \ 4\ \ 0\ \ 0\ \ 0)$ and

$c_3^T = (3\ \ 7\ \ 1\ \ 5\ \ 5\ \ 6\ \ 0\ \ 0\ \ 0)$

$r_7 = (0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 9\ \ 7)$ and

$c_8^T = (0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 9\ \ 1\ \ 5)$

Step 5: The scalar products b are

$b_1 = r_1 \cdot c_3^T = (3\ \ 28\ \ 3\ \ 5\ \ 15\ \ 24\ \ 0\ \ 0\ \ 0)$

$b_2 = r_7 \cdot c_8^T = (0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 9\ \ 9\ \ 35)$

Step 6: The bias identifying vectors f are

$f_1 = r_1 \cdot c_3^T - b_{13} = (0\ \ 25\ \ 0\ \ 2\ \ 12\ \ 21\ \ -3\ \ -3\ \ -3)$

$f_2 = r_7 \cdot c_8^T - b_{78} = (-9\ \ -9\ \ -9\ \ -9\ \ -9\ \ -9\ \ 0\ \ 0\ \ 26)$

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Step 7: From the scalar products b_1 and b_2 and the bias identifying vectors f_1 and f_2, we can find that the inconsistent values are $b_{12}b_{23}=28$ and $b_{79}b_{98}=35$, which are corresponding the inconsistent elements $A_{12}A_{23}=28$ and $C3_{13}C3_{32}=35$. Besides, since there are three elements in f_1 far away from b_{13}, therefore, the corresponding a_{13} might be too small.

Furthermore, $c_{12}=3.7619>0$, $c_{23}=-5.25<0$, $c_{79}=-5.2<0$, $c_{98}=-3.7143<0$, and $c_{13}=60>0$. These inequalities indicate that the value of b_{12} is too small; the value of b_{23} is too large; and the values of b_{79} and b_{98} are probably too large, respectively. Besides, the value of $b_{79}b_{98}$ is supposed to be 9 or close to 9 instead of 35. That is, the value of a_{12} is small; the value of a_{23} is too large; the values of $c3_{13}$and $c3_{32}$ are possible too large and a_{13} is too small. Those elements should be revised. For instance, Let

$$a_{13}=9,\ a_{31}=\tfrac{1}{9},\ a_{23}=3,\ a_{32}=\tfrac{1}{3}\ c3_{13}=2,\ c3_{31}=\tfrac{1}{2},\ c3_{32}=4,\ c3_{23}=\tfrac{1}{4}$$

The values of the elements in the sub-matrix A and sub-matrix $C3$ are replaced with the above corresponding values of elements and calculate their maximum eigenvalues to test the consistency. That is, the maximum eigenvalues are $\lambda^0{}_{\max}=7.1374$ and$\lambda^4{}_{\max}=3.0246$.

Test the consistencies using the maximum eigenvalue threshold method:

$$\lambda^0_{\max}=\lambda^0_{\max}-\lambda^6_{\max\ \text{thrd}}=7.1374-6.781=0.3564>0$$
$$\Delta\lambda^4_{\max}=\lambda^3_{\max}-\lambda^3_{\max\ \text{thrd}}=3.0246-3.104=-0.0794<0$$

The comparison matrix $C4$ passed the consistency test, however, the comparison matrix A is still inconsistent. Identify the inconsistent elements in the comparison matrix A using the second largest bias value 39, the third largest bias value 23.6, and the fourth largest bias value 21.3333. These three elements are located at 5th row and 3rd column, 1st row and 4th column, and 5th row and 4th column, respectively. Repeating the above steps, the corresponding bias identifying vector $f_i\,(i=1,2,3)$ becomes

$$f_1=r_5\cdot c_3^T-b_{53}=\begin{pmatrix}-4 & 30 & 0 & 0 & 0 & 13 & -5 & -5 & -5\end{pmatrix}$$
$$f_2=r_1\cdot c_4^T-b_{14}=\begin{pmatrix}0 & 11 & -0.4 & 0 & 2 & 11 & -1 & -1 & -1\end{pmatrix}$$
$$f_3=r_5\cdot c_4^T-b_{54}=\begin{pmatrix}-0.6667 & 14 & 0 & 0 & 0 & 8 & -1 & -1 & -1\end{pmatrix}$$

From the above three bias identifying vectors f_1, f_2, and f_3, we can find that the inconsistent values are

In f_1 : $b_{52}b_{23}=35$ and $b_{56}b_{63}=18 \Leftrightarrow a_{52}a_{23}=35$ and $a_{56}a_{63}=18$ in matrix A

In f_2 : $b_{12}b_{24}=11$ and $b_{16}b_{64}=11 \Leftrightarrow a_{12}a_{24}=11$ and $a_{16}a_{64}=11$ in matrix A

In f_3 : $b_{52}b_{24}=14$ and $b_{56}b_{64}=8 \Leftrightarrow a_{52}a_{24}=14$ and $a_{56}a_{64}=8$ in matrix A

where the symbol '$\Leftrightarrow$' denotes 'corresponding to'

The following inequalities show the inconsistent elements:

$c_{52}=-14.619<0 \Rightarrow a_{52}$ is too large

$c_{23}=-5.25<0 \Rightarrow a_{23}$ is too large

$c_{56}=-4.5<0 \Rightarrow a_{56}$ is too large

$c_{63}=0.41675>0 \Rightarrow a_{63}$ is slightly small

$c_{12}=3.7619>0 \Rightarrow a_{12}$ is too small

$c_{24}=-7.15<0 \Rightarrow a_{24}$ is too large

$c_{16}=-2.1667<0 \Rightarrow a_{16}$ is too large

$c_{64}=-7.2167<0 \Rightarrow a_{64}$ is too large

For instance, decrease the values of the three elements with the largest absolute values a_{52}, a_{24}, and a_{64}. Let

$$a_{52}=1,\ a_{25}=1;\ a_{24}=1/2,\ a_{42}=2;\ a_{64}=1/2,\ a_{64}=2$$

Replacing all the corresponding values in the original comparison matrix A with the above values and calculating the maximum value of matrix A, we can get $\lambda^0_{\max}=6.6156<\lambda^6_{\max\ \text{thrd}}=6.781$. The comparison matrix passes the consistency test. If the inconsistent elements identified in the first time, a_{13} and a_{23}, are replaced with $a_{13}=9, a_{31}=1/9$, $a_{23}=3, a_{32}=1/3$, and continue to calculate the maximum eigenvalue of comparison matrix A, then$\lambda^0_{\max}=6.3174<\lambda^6_{\max\ \text{thrd}}=6.781$. The comparison matrix A passes the consistency test with smaller maximum eigenvalue, which is corresponding to the small CR.

5. Conclusions

The ANP, as one of the widely used MCDM method, can be applied to assess the key factors of the risk and the potential risk, determine risk level and risk consequences, and analyze the uncertain variables and preferences of a decision. However the consistency test index and the inconsistency identification method need to be simplified. In this paper, a maximum eigenvalue threshold index is proposed as a new consistency index for the ANP when the ANP is applied to risk assessment and decision analysis, because the consistency ratio (CR) method is quite complicated for the ANP, especially in the case of risk assessment and decision analysis of an emergent event. The proposed consistency index is mathematically equivalent to the CR method, but easier and simpler than the CR method in practical as the consistency index for the ANP, which is a requisite for the ANP when being applied in risk assessment and decision analysis. In addition, a block diagonal matrix is introduced to simplify the process and conduct the inconsistency tests of several comparison matrices all together. Besides, an induced bias block diagonal comparison matrix is proposed to further identify the inconsistent elements, and two identification principals, gradual-level identification principal and whole-level identification principal, are also proposed to further identify the inconsistent elements properly. The two illustrative examples of the emergency decision making show the effectiveness and the simplicity of the proposed consistency index, as well as the inconsistency identification and adjustment method.

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Although the inconsistent comparison matrices can be identified easily using the proposed index, and the inconsistent elements in the comparison matrices also can be identified and adjusted using the identification method, the simplest inconsistency identification for the higher and much more complicated comparison matrices still remains to be studied; the extensive application of the ANP in risk assessment and decision analysis is also another research direction in the future.

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