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Steven Nahmias

Perishable Inventory Systems

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For Bill Pierskalla, who opened the door

Preface

Inventory control has emerged as a leading application of operations research. The Survey of Current Business reported that the dollar value of inventories in the USA alone exceeded \$1.3 trillion at the end of 2010. Cost-effective control of inventories can cut costs significantly, and contribute to the efficient flow of goods and services in the economy. Many techniques can be brought to bear on the inventory management problem. Linear and nonlinear programing, queueing, and network flow models, are some examples. However, most inventory control packages are based on the methodology of inventory theory. Inventory theory is an important subfield of operations research that addresses the specific questions: when should an order be placed, and for how much?

Inventory theory had its roots in the well-known EOQ formula, first discovered by Ford Harris nearly 100 years ago (Harris 1915). Harris, working as a young engineer at the Westinghouse Corporation in Pittsburgh, was able to see that a simple formula for an optimal production batch size could be obtained by properly balancing holding and set-up costs. The EOQ formula, first derived by Harris, is amazingly robust – it still serves as an effective approximation for much more complex models. After Harris's work, the development of inventory theory was largely stalled until after World War II. The success of operations research in supporting the war effort was the spur needed to get the field off the ground. It seems that the newsvendor model of inventory choice under uncertainty was developed around this time, although it appears that the fundamental approach of balancing overage and underage costs under uncertainty was really first derived by Edgeworth (1888) in the context of banking.

Serious research into stochastic inventory models began around 1950. An early landmark paper was Arrow, Harris, and Marschak (1951). They were the first researchers to provide a rigorous analysis of a multiperiod stochastic inventory problem. Three significant books on the theory stimulated substantial interest in inventory theory research: Whitin (1957), Arrow, Karlin, and Scarf (1958), and Hadley and Whitin (1963). The 1960s saw an explosion of papers in inventory theory.

None of the books or hundreds of papers on inventory control written up to this time addressed an important class of problems. In every case, a tacit assumption was made that items stored in inventory had an infinite lifetime and unchanging utility. That is, once placed into stock, items would continue to have the same value in the marketplace in perpetuity. In truth, there is a very large class of inventories for which this assumption is wrong. These include inventories subject to decay, obsolescence, or perishability.

Let us define our terms. Decay (or exponential decay) means that a fixed fraction of the inventory is lost every planning period (this has also been referred to as age independent perishability). In continuous time, this translates to the size of the inventory decreasing at an exponential rate. Very few real systems are accurately described by exponential decay. For example, suppose the local grocery store discards an average of 10% of its production each day due to spoilage. In actuality though, some days it will not have to discard any product and some days it will have to discard much more than 10%. Assuming a 10% loss each day is a convenient approximation of a more complex process. Exponential decay has been proposed as a model for evaporation of volatile liquids, such as alcohol and gasoline. But how often are these substances stored in open containers, so that they would be subject to evaporation? Radioactive substances (such as radioactive drugs) are one example of true exponential decay. However, inventory management of radioactive substances is a rather specialized narrow problem. While exponential decay has been proposed as an approximation for fixed life perishables, there are better approximations.

A related problem is that of managing inventory subject to obsolescence. What distinguishes obsolescence from perishability is the following. Obsolescence typically occurs when an item has been superseded by a better version. Electronic components, maps, and cameras are examples of items that become obsolete. Notice that in each case, the items themselves do not change. What changes is the environment around them. As a result of the changing environment, the utility of the item has declined. In some cases, the utility goes to zero, and unsold items are salvaged or discarded. However, it is often the case that utility does not decrease to zero. Declining utility can result in declining demand and/or decreasing prices. For example, older electronic items, such as a prior generation of PDAs or hard drives, continue to be available for some time, but are typically sold at reduced prices. From a modeling perspective, the point at which an item becomes obsolete cannot be predicted in advance. Hence, obsolescence is characterized by uncertainty in the useful lifetime of the product.

Finally, we come to perishability. We assume the following definition of perishability throughout this monograph. A perishable item is one that has constant utility up until an expiration date (which may be known or uncertain), at which point the utility drops to zero. This includes many types of packaged foods, such as milk, cheese, processed meats, and canned goods. It also includes virtually all pharmaceuticals and photographic film. This writer's interest in this area was originally sparked by blood bank management. Whole blood has a legal lifetime of 21 days, after which time it must be discarded due to the buildup of contaminants. When uncertainty of the product lifetime is assumed, the class of items one can model is substantially larger. For example, perishable inventory with an uncertain lifetime can accurately describe many types of obsolescence.

Considering the large number of perishable items in the economy, why was this important class of problems ignored for so long? The short answer is that the problems are difficult to analyze. Interestingly, Pete Veinott, a major figure in inventory theory, wrote his doctoral thesis (in the early 1960s) on various deterministic models for ordering and issuing perishable inventories, but never published this work. When this writer inquired why, he said that the notation was so complex and awkward, and he preferred putting the work aside and move on to other problems (Veinott 1978). Van Zyl's (1964) important work on the two period lifetime case with uncertain demand remained largely unknown, as it was never published in the open literature. (This author became aware of Van Zyl's work after completing his doctoral thesis on the subject).

Contents

Chapter 1
Preliminaries

Contents

1.1 Deterministic Demand

When demand is known with certainty, the problem of managing perishables is straightforward for the most part. Consider first the basic EOQ model. Suppose the demand rate is λ, the fixed cost of placing new orders is K, and the holding cost per unit time is h. Then, it is well known that the optimal order size is

$$Q^* = \sqrt{\frac{2K\lambda}{h}}$$

and the optimal time between placement of orders is $T^* = Q^*/\lambda$. Suppose now that the item has a usable lifetime of m. All deliveries are assumed to be of fresh units only. Then, there are two cases: (a) $T^* \leq m$ and (b) $T^* > m$. In case (a) the optimal policy remains the same, since in each order cycle, all units are consumed by demand before they expire. However, in case (b) if Q^* is ordered at the beginning of the cycle, there will be positive inventory on hand at time m, which will have outdated and must be discarded, and a new order placed at that time. Notice, however, that if we reduce the order quantity from $Q^* = \lambda T^*$ to $Q = \lambda m < Q^*$, then the cycle length will remain at m, no units will expire and holding costs will be reduced, since average inventory will be reduced from $Q^*/2$ to $Q/2$. Hence, the modification of the standard EOQ model to include perishability is straightforward.

However, not all deterministic perishable inventory problems are solved so easily. In particular, consider the deterministic nonstationary production planning

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problem. Demands over an n period planning horizon are known constants, say $(r_1, r_2, \ldots, r_n)$. Costs include holding, h_i, set-up, K_i, and marginal production cost, C_i in each period. Then, Wagner and Whitin (1958) showed (in the infinite lifetime case) that an optimal policy has the following structure. If starting inventory is zero, then the order quantity in each period is either zero or exact requirements – namely, the sum of requirements in the current to some future period. Furthermore, an optimal policy only orders in periods when the starting inventory is zero. As a consequence of this result, one only needs to determine the periods in which ordering takes place, thus reducing the calculations significantly.

It turns out that an exact requirements policy may not be optimal when perishability is introduced. A counterexample is presented in Chap. 7, and methods for resolving this problem is discussed there. When demand is nonstationary, finding optimal order policies for a fixed life inventory is not trivial. However, the vast majority of the research on ordering policies for perishables has focused on stochastic demand – a significantly more difficult problem.

1.2 Periodic Review Versus Continuous Review

Demand uncertainty and (fixed life) perishability combine to result in challenging and complex problems. Stochastic perishable inventory problems fall into one of the two basic categories: periodic review or continuous review.

Most of the research in inventory theory assumes inventory levels are reviewed periodically. This means that the state of the system (on hand inventory) is known only at discrete points in time. This assumption is appropriate, for example, if inventory levels are checked once a day, once a week, etc. The landmark collection of Arrow et al. (1958) assumed periodic review in every case considered, and set the stage for much of the subsequent research on inventories. From a practice point of view, it is probably true that most inventory systems were periodic review 50 years ago. Today, however, point-of-sale scanners and automated inventory control systems have made true continuous review more common.

There are two reasons why continuous review has grown in importance. First, with automated inventory control systems computers can automatically trigger orders when inventory levels hit predetermined levels. Second, continuous review models often are able to provide simple approximations to complex problems that are difficult to solve with periodic review formulations.

Perishable inventory research has also evolved along the two separate tracks of periodic review and continuous review. The periodic review track generalizes the kind of models considered by Arrow et al. (1958), among many others, to incorporate perishability. The continuous review track is largely an outgrowth of the theory of queues with impatient servers. An impatient customer is one who leaves the queue if they have not been served by a fixed time. Queueing models with impatient customers are discussed in detail in Chap. 9.

1.3 Periodic Review Preliminaries

As noted earlier, we assume that the on hand inventory level is known only at discrete points in time, which are labeled periods. Assume that periods are numbered 1, 2,. . .. Demands in successive periods are not known, but are assumed to be random variables, $D_1, D_2, \ldots$ with a known probability distribution. For convenience, assume that the demand distribution is continuous with cumulative distribution function (CDF) $F(x)$ and probability density function (PDF) $f(x)$. (Note that basic results have been shown to carry over to the discrete demand case as well. Also, virtually all of the results carry over to nonstationary demand. Stationarity of the demand distribution is assumed for notational convenience.)

Assume that new orders are always of fresh units that have a usable lifetime of m periods. A little reflection should convince the reader that it is necessary to track the entire age distribution of the on hand inventory in order to determine outdates each period. Hence, the system state is described by a vector $\mathbf{x} = (x_{m-1}, x_{m-2}, \ldots, x_1)$ where x_i is the number of units on hand with i useful periods of life remaining. Note that there are many notation options for the state vector. The state could be defined in terms of age rather than remaining lifetime and the vector could be numbered in order of oldest to youngest rather than vice versa, as we have done. This convention was chosen to reflect that aging occurs in the direction left to right, like the flow of the English language, and that the decision variable, y, can be equated to x_m and placed in the proper position in the vector. Note that x_0 would represent the number of units on hand that have just expired or outdated. We do not need to carry x_0 in the state vector since outdated units are assumed to leave the system. We use the convention throughout that boldface $\mathbf{x}$ is the vector of on hand inventories of each age level, and $x = \sum_{i=1}^{m-1} x_i$ is a scalar quantity representing the total on hand inventory.

The necessity to define a vector valued state variable is only one of the things that separate the perishable inventory problem from the conventional nonperishable problem. As we see, several other concerns arise as a result of perishability. One is the sequence that items are issued to meet demand. Note that there is a substantial literature on optimal issuing policies independent of the ordering problem. The appropriate assumption concerning issuing policies depends on whether the producer or consumer chooses which items satisfy demand. If the producer determines the issuing policy, it is clearly in his interest to issue items on an oldest first basis (known in accounting parlance as FIFO for first in first out). If the consumer determines the issuing policy, it is likely that the consumer will choose the freshest items, resulting in units issued in last in first out (LIFO) sequence. The vast majority of the perishable inventory literature assumes FIFO issuing, and we do so as well unless stated otherwise. Clearly, FIFO is most cost-efficient and results in minimum outdating. (A third alternative, which might be appropriate in some contexts, is to issue the items in a random order. To our knowledge, random issuing policies have not been considered in the context of optimal ordering policies for perishables.)

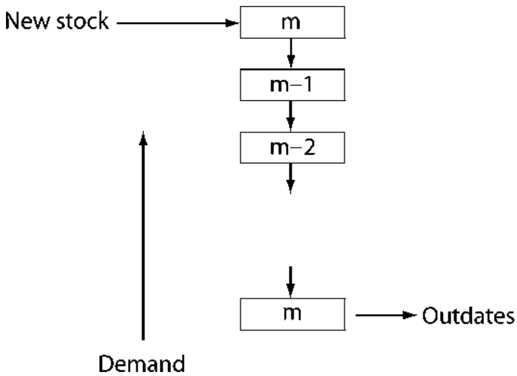

Fig. 1.1 The flow of demand and product in an FIFO fixed life perishable inventory system

If items are issued according to FIFO, then the aging and demand processes travel in opposite directions. To see what this means, consider the representation of the system state in Fig. 1.1. Bins are labeled $m-1$, $m-2$, . . ., 1, where the contents of bin i are the number of units on hand with i useful periods of life remaining. At the end of each period, all contents of a bin are moved to the next lower bin, and the contents of bin 1 outdate and must be discarded (or salvaged). Because of the FIFO assumption, demand depletes first from bin 1, then from bin 2, etc. Excess demands may be lost or backordered. If excess demands are backordered, then this is reflected in a negative value of x_{m-1}.

Consider now the ordering policy. The optimal number of units to order each period is function of the state vector, $\mathbf{x}$. We represent this function as $y(\mathbf{x})$. As we see, $y(\mathbf{x})$ is a complex nonlinear function of the state variable x. We assume that costs are assessed in the usual way for finite horizon periodic review inventory systems. At the end of each period, the total inventory is determined. If it is positive, assess a holding cost of h per unit held per period. If it is negative (which occurs when excess demands are backordered), assess a cost of p per unit of unsatisfied demand. Furthermore, we assume a marginal order cost only. That is, there is a cost of c per unit ordered. For now, assume that there is no fixed order cost. Finally, we come to the issue of how to assess the cost for items that must be discarded due to outdating. Let θ be the cost of disposing of outdated units. If D is the demand in a period, then the number of units outdating at the end of the period when starting inventories are $\mathbf{x}$, is $\max(x_1 - D, 0)$, which we represent as $(x_1 - D)^+$.

We now face the first issue. The astute reader will notice that the outdating cost, $\theta E(x_1 - D)^+$, is independent of the decision variable y. That means that any single period model ignores the effects of outdating. In fact, one would need to churn through at least m periods of a dynamic programing formulation before the outdating penalties of over ordering would be reflected in the optimal order policy. (This was, in fact, the approach taken in Fries 1975.)

Suppose, however, that one were interested in constructing a one period model that reflected the outdating penalties of over ordering. How could this be done? Let $D_1, D_2, \ldots$ represent demands in successive periods, starting with the current period. Then, the current order, y, would not outdate until m periods into the future, if it had not been consumed by demand by that time. Consider how one would determine the expected outdating of the current order y, m periods into the future.

Define the following sequence of random variables:

$$R_1 = y + \sum_{i=1}^{m-1} x_i$$
$$R_2 = [y + \sum_{i=2}^{m-1} x_i - (D_1 - x_1)^+]^+$$
$$R_3 = [y + \sum_{i=3}^{m-1} x_i - (D_2 + (D_1 - x_1)^+ - x_2)^+]^+$$

etc.

Then, R_i represents the amount of the current inventory on hand i periods into the future, assuming we start with $\mathbf{x}$ and order y. Each value of R_i is a random variable, since it is a function of the future demands. Note the use of the imbedded + functions are necessary to keep track of units lost due to outdating.

To simplify the notation, define the following sequence of random variables recursively:

$$\begin{aligned} B_0 &= 0 \\ B_1 &= (D_1 - x_1)^+ \\ &\vdots \\ B_j &= (D_j + B_{j-1} - x_j)^+ \text{ for } 1 \le j \le m-1. \end{aligned}$$

Interpret the random variable B_j as the total unsatisfied demand in period j after depleting the on hand inventory that would have outdated in period j. It follows that

$$R_m = [y - (D_m + B_{m-1})]^+$$

represents the amount of the current order y, that outdates in m periods.

The goal of the analysis is to compute the expected value of R_m and incorporate this into the one period model.

Define $G_n(t;\, \mathbf{w}_{n-1}) = P\{D_n + B_{n-1} \le t)$ where $\mathbf{w}_i = (x_i, x_{i-1}, \ldots, x_1)$. Note that $\mathbf{w}_{m-1} = \mathbf{x}$.

We present the first result without proof, which is based on a standard induction argument. Details can be found in Nahmias (1972).

Theorem 1.1. $G_n(t;\, \mathbf{w}_{n-1}) = \int_0^t G_{n-1}(v + x_{n-1};\, \mathbf{w}_{n-2}) f(t-v)\mathrm{d}v.$

Theorem 1.2. $E(R_m) = \int_0^y G_m(t; \mathbf{x})\mathrm{d}t$

Proof. It is well known that for any nonnegative random variable, X, the expectation may be computed two ways:

$$E(X) = \int_0^\infty x f(x)\mathrm{d}x = \int_0^\infty [1 - F(x)]\mathrm{d}x.$$

We have $P\{R_m \leq t\} = P\{y - (D_m + B_{m-1}) \leq t\} = 1 - G_m(y - t;\, \mathbf{x})$ for $t \geq 0$ Since R_m is a nonnegative random variable, the result follows from the second representation of the expected value above (after a change of variable). $\square$

1.4 A One Period Newsvendor Perishable Inventory Model

Most readers should be familiar with the classic newsvendor model. A newsvendor must decide at the beginning of each day how many newspapers to purchase. Daily demand is not known, but is assumed to follow a known probability distribution. Let y be the number of newspapers purchased and D the demand. There are two penalties: overage (ordering too much) and underage (ordering too little).

Now, let us consider the perishable inventory model. The penalty for ordering too much is the future penalty of outdating, at θ per unit, and the penalty for ordering too little is penalty cost for excess demand, at p per unit. Hence, a sensible expected one period cost function for the perishable inventory problem is:

$$L(\mathbf{x}, y) = p \int_{x+y}^{\infty} [t - (x + y)]\, f(t)\mathrm{d}t + \theta \int_{0}^{y} G_m(t; \mathbf{x})\mathrm{d}t.$$

It is easy to show that $L(\mathbf{x}, y)$ is convex in y (and is strictly convex as long $f(t) > 0$ for all $t > 0$. Hence, the optimal order quantity, y, for this simple model satisfies:

$$\frac{\partial L(\mathbf{x}, y)}{\partial y} = -p(1 - F(x + y)) + \theta G_m(y;\, \mathbf{x}) = 0.$$

The optimal one period solution, say $y^*(\mathbf{x})$, is a nonlinear function of the entire state vector, $\mathbf{x}$. In this case, $y^*(\mathbf{x}) > 0$ for all positive real vectors $\mathbf{x}$. In addition, as we see in the analysis of the dynamic problem, $y^*(\mathbf{x})$ is decreasing in each component of the state vector, $\mathbf{x}$, but at less than unit rate.

Somewhat sharper results can be obtained when we add holding and marginal order costs.

Theorem 1.3. Suppose that in addition to penalty and outdate costs, we also include marginal order cost at c per unit ordered, and a unit holding cost, h, charged against each unit on hand at the end of the period. Then, the optimal solution has the following form: If $x < \bar{x}$ order $y^*(\mathbf{x})$ solving

$$c + hF(x + y) + p(1 - F(x + y)) + \theta G_m(y;\, \mathbf{x}) = 0$$

where $\bar{x}$ solves

$$c + hF(\bar{x}) - p(1 - F(\bar{x})) = 0.$$

If $x \geq \bar{x}$, no order is placed.